STOCHASTIC INSTABILITY OF NONLINEAR OSCILLATIONS

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Consider a dynamic system whose behavior is described by

$$x^{\cdot\cdot} + \omega^2{}_0(1 + \alpha x^2)\, x + F(t)x = 0 \qquad (0.1)$$

in which the external force F(t) is very nearly periodic, with frequency $\Omega \ll \omega_0$. It is known [1,2] that such a system retains its periodic motion if the nonlinearity parameter α is sufficiently small, and such motion will be called stable. Let us examine the conditions under which system (0.1) is a dynamic system with mixing in the sense of [3,4], i.e., what are the conditions under which system (0.1) may be described approximately by means of statistical laws (the stochastic criterion, SC). The solution to (0.1) may be represented as quasi-periodic motion with a substantially varying phase. The criterion is deduced from the condition that the time correlations of the phase decrease exponentially. Roughly speaking, we must be able to state the moment at which the time sequence of the phases can be considered approximately as a sequence of random numbers. It is clear that in that case we speak of stochastic motion, not in the whole of phase space, but in a region where there is ergodic motion with respect to phase.

There are major difficulties in defining a criterion for stochastic behavior; this criterion has recently [5–8]* been derived for some very simple physical systems. The titles of these papers show the importance of the topic. An equation of the type (0.1) has also been used in examining the stability of magnetic field lines in systems of closed type.

Here we give an asymptotic method of defining the SC for motion of the type (0.1), which allows extension to somewhat more complicated systems. The method cannot be said to be mathematically rigorous; certain points need a more rigorous basis, although they are entirely reasonable from the physical viewpoint.

§ 1. Parametric force with instantaneous action. Consider first the case in which the parametric force F(t) has characteristic periodic discontinuities:

$$x^{\cdot\cdot} + \omega^2{}_0 (1 + \alpha x^2)\, x + a \sum \delta (t - t_k)\, x = 0\,. \qquad (1.1)$$

The jumps at t_k follow at equal intervals $T = 1/\Omega$. Between two such jumps, x is the solution to the equation

$$x^{\cdot\cdot} + \omega_0{}^2 (1 + \alpha x^2)x = 0\,. \qquad (1.2)$$

This equation has been studied in detail for

$$\alpha x^2 \ll 1\,. \qquad (1.3)$$

However, it is convenient to represent the solution to (1.2) in asymptotic-series WKB-approximation form [9]; using (1.3), we write [8]

$$x_\pm = \omega^{-1/2} \exp\left\{\pm\, i \int^t \omega\,(t')\, dt'\right\}$$

$$(\omega = \omega_0 \sqrt{1 + \alpha x^2})\,. \qquad (1.4)$$

We assume that subsequently some approximate expression for x(t) will be substituted as the expression for ω in (1.4). For instance, in zeroth approximation

$$\omega = \omega_0 \sqrt{1 + \alpha x_0{}^2}, \qquad x_0 = A_0 \cos \omega_0 t\,. \qquad (1.5)$$

In fact, (1.5) is not really correct, since, if we substitute in (1.2) for the frequency in accordance with (1.5), we get parametric resonance, which in fact does not occur in (1.2). This difficulty is easily eliminated if in the definition of x_0 we use for the frequency a value, corrected in accordance with the usual theory [10]:

$$x_0 = A_0 \cos\, (\omega_0 + \Delta\omega)t, \qquad \Delta\omega = {}^3/_8 \alpha A_0{}^2\,. \qquad (1.6)$$

In what follows it is assumed that x_0 is defined by (1.6) in formula (1.5) for ω.

A second comment on the solution of (1.4) arises from the inequality

$$\omega \gg \Omega \qquad (1.7)$$

for which (1.1) is to be examined; according to (1.7), between two successive jumps there will be numerous points Re t* such that $\omega(t^*) = 0$. Near the points t*, the asymptotic solutions of (1.4) become inapplicable because of the Stokes effect. Logical allowance for all the singularities t* leads us to multiply the solutions of (1.4) by a periodic factor that varies slowly in time. This effect will be neglected if the effect of the jumps in F(t) is important (the corresponding estimate is given in Appendix 1). We write the solution of a type similar to (1.4) in the following form between two successive steps at t_{n-1} and t_n:

$$x_\pm{}^{(n)}\, t \approx \omega_n{}^{-1/2} \exp\left\{\pm\, i \int_{t_n}^{t} \omega_n\,(t')\, dt'\right\},$$

$$\omega_n = \omega_0 \sqrt{1 + \alpha x_n{}^2}, \qquad x_n = A_n x_+{}^{(n)} + \bar{A}_n x_-{}^{(n)}, \quad (1.8)$$

in which the bar denotes the complex conjugate. The continuity of the solution at the points t_n is used with (1.1) to relate A_n and $\bar{A}_n$ to A_{n+1} and $\bar{A}_{n+1}$:

$$\begin{pmatrix} A_{n+1} \\ \bar{A}_{n+1} \end{pmatrix} = \frac{1}{2}\left(\frac{\omega_{n+1}}{\omega_n}\right)^{1/2} \times$$

$$\times \begin{pmatrix} (\beta_n{}^+ + i\varepsilon_n) \exp iS_{n+1} & (\beta_n{}^- + i\varepsilon_n) \exp iS_{n+1} \\ (\beta_n{}^- - i\varepsilon_n) \exp(-iS_{n+1}) & (\beta_n{}^+ - i\varepsilon_n) \exp(-iS_{n+1}) \end{pmatrix} \times$$

$$\times \begin{pmatrix} A_n \\ \bar{A}_n \end{pmatrix};$$

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*See also B. V. Chirikov, Dissertation, Novosibirsk, 1959.

$$\varepsilon_n = \frac{a}{\omega_{n+1}},$$

$$S_{n+1} = \int_{t_n}^{t_{n+1}} \omega_{n+1}(t')\,dt' = \int_{t_n}^{t_n+T} \omega_{n+1}(t')\,dt',$$

$$\beta_n^{\pm} = 1 \pm \frac{\omega_n}{\omega_{n+1}}. \qquad (1.9)$$

It is convenient to put (1.9) in the condensed form

$$\mathbf{A}_{n+1} = \mathbf{M}_n \mathbf{A}_n. \qquad (1.10)$$

The matrix M is equivalent to an operator, acting on x(t), for displacement by T. These transformations consist in replacing the initial equation (1.1) by the equivalent equations (1.9) or (1.10) in finite differences.

From (1.9) we get

$$\operatorname{tg}(\varphi_{n+1} - S_{n+1}) = \frac{\omega_n}{\omega_{n+1}} \operatorname{tg}\varphi_n + \varepsilon_n \left|\frac{A_{n+1}}{A_n}\right|^2 + \frac{\omega_{n+1}}{\omega_n}\Big(1 - \beta_n^- \sin^2\varphi_n + \frac{\omega_n}{\omega_{n+1}} \varepsilon_n \sin 2\varphi_n + \varepsilon_n^2 \cos^2\varphi_n\Big),$$

$$A_n = |A_n| \exp i\varphi_n, \qquad A_{n+1} = |A_{n+1}| \exp i\varphi_{n+1}. \qquad (1.11)$$

Formulas (1.9)–(1.11) can be used only if there is some upper bound to the magnitude a of a jump. Condition (1.3) gives from (1.11) that

$$\alpha x_n^2 \max(\varepsilon_n, \varepsilon_n^2) \ll 1, \qquad (1.12)$$

which means that the nonlinearity remains small throughout the process and that it is meaningful to represent the solutions to (1.1) in the form (1.8).

Equation (1.11) gives the following relation between the phases* φ_n and φ_{n+1}:

$$\varphi_{n+1} = \left\{S_{n+1} + \operatorname{arc\,tg}\left(\frac{\omega_n}{\omega_{n+1}} \operatorname{tg}\varphi_n + \varepsilon_n\right)\right\}, \qquad (1.13)$$

in which $\{\ldots\}$ denotes the fractional part.

Evaluation of S_{n+1} and use of (1.7) replaces (1.13) by

$$\varphi_{n+1} \approx \left\{\omega_0 T\left(1 + \frac{\alpha}{2\omega_0}|A_{n+1}|^2\right) + \operatorname{arc\,tg}\left(\frac{\omega_n}{\omega_{n+1}} \operatorname{tg}\varphi_n + \varepsilon_n\right)\right\}, \qquad (1.14)$$

which may be put as

$$\varphi_{n+1} = \{k f(\varphi_n)\}, \qquad (1.15)$$

where k is large and so $kf(\varphi_n) \gg 1$ for almost all φ_n apart from the small region $\Delta\varphi \sim k^{-1}$.

*The phases are normalized in such a way that the changes occur within the range of Eq. (0.1).

We now examine (1.15). It has been shown [11] that the dynamic system described by

$$\varphi_{n+1} = \{k\varphi_n\}, \qquad k \gg 1 \qquad (1.16)$$

is a system with mixing, i.e., φ_n may be considered as random with a certain degree of accuracy. This feature may be illustrated by the phase correlation

$$R(1) = \left(\int_0^1 \left(\varphi_{n+1} - \frac{1}{2}\right)\left(\varphi_n - \frac{1}{2}\right) d\varphi_n\right)\left[\int_0^1 \left(\varphi_n - \frac{1}{2}\right)^2 d\varphi_n\right]^{-1}. \qquad (1.17)$$

The integral of (1.17) has been calculated [6] for an equation in a form analogous to (1.16). For $k \gg 1$ we have

$$R(1) \approx C_1 / k, \qquad (1.18)$$

in which $C_1 \sim 1$. It is readily shown that

$$R(m) = \int_0^1 \left(\varphi_{n+m} - \frac{1}{2}\right)\left(\varphi_n - \frac{1}{2}\right) d\varphi_n \times \left[\int_0^1 \left(\varphi_n - \frac{1}{2}\right)^2 d\varphi_n\right]^{-1} \approx \frac{C_m}{k^m}. \qquad (1.19)$$

where $C_m \sim 1$. Equations (1.18) and (1.19) mean that generally

$$R(n) \approx C(n) e^{-n \ln k}, \qquad (1.20)$$

in which C(n) is a function of n that varies slowly relative to the exponential factor. The latter is a consequence of the condition $k \gg 1$. The result of (1.20) implies exponential decay of the phase correlation, so $k \gg 1$ for (1.16) may* be considered as the stochastic criterion.

A similar analysis is possible for (1.15). We substitute (1.14) into (1.17) to get

$$R(1) \sim K^{-1},$$

$$K = \max(a/\omega,\ a^2/\omega^2)\alpha G^2 \omega / \Omega \gg 1, \qquad (1.21)$$

in which G is the oscillation amplitude; this is the SC for the phase of the nonlinear oscillator described by (1.1). There is a region $\Delta\varphi_n \sim K^{-1}$ in which the criterion is not obeyed, and so a more rigorous analysis requires an evaluation of the time spent by the system in such regions (see Appendix 2). Condition (1.21) may be put in the usual form

$$R(\tau) = \frac{\langle(\varphi(t) - 1/2)(\varphi(t+\tau) - 1/2)\rangle}{\langle(\varphi(t) - 1/2)^2\rangle} \sim \exp\frac{-\tau}{\tau_0},$$

$$\tau_0 = \frac{T}{\ln K} = (\Omega \ln K)^{-1} \qquad (1.22)$$

*In fact, it has been shown [11] that $k > 1$ suffices.

in which $\langle \ldots \rangle$ denotes averaging over the space of $\varphi(t)$.

Condition (1.21) is ensured by the great distance between jumps and thus by the possibility for a large phase change between them. For a given large ω/Ω, reduction in α and in a/ω leads to violation of (1.21), in accordance with Kolmogorov's theorem on the stability of motion [1].

§ 2. Adiabatic parametric force. Consider the case of an F(t) in (1.1) that is in a certain sense the converse of the previous case §1:

$$x^{\cdot\cdot} + \omega_0^2 (1 + \alpha x^2)x - V \cos \Omega t x = 0 , \qquad (2.1)$$

where $\omega_0 \gg \Omega$. Then the oscillator is under adiabatic conditions, and the change in its adiabatic invariant is exponentially small. This does not allow us to obtain a large phase change and corresponds to k always small in an equation similar in form to (1.15), except when $V \sim \omega_0^2$, when (as is well-known for a linear oscillator) the relative change in the adiabatic invariant is on the order of unity. We therefore naturally consider the region

$$0 < \omega_0^2 - V \ll \omega_0^2 \qquad (2.2)$$

in more detail. In order to be able to consider the nonlinearity small, we put

$$\alpha x^2 \ll (\omega_0^2 - V) / \omega_0^2 , \qquad (2.3)$$

which enables us to put the solution to (2.1) in a form analogous to (1.8) via the substitution

$$\omega_n(t) = \sqrt{\omega_0^2 (1 + \alpha x^2_n) - V \cos \Omega t} . \qquad (2.4)$$

The points $\omega_n(t) = 0$ are singular points in the solution of (1.8). The main contribution to the change in the adiabatic invariant comes from those points t_0 in the complex plane of t that lie nearest to the real axis. From (2.2), $\operatorname{Re} t_0 \sim 2\pi/\Omega$, and the points $\operatorname{Re} t_0$ can be interpreted as jumps by analogy with the case considered in the previous section. Matrix M is similar to that of (1.9) with the substitution

$$\varepsilon_n = e^{-\delta}, \qquad \delta \approx \frac{\pi}{2} \frac{\omega_0^2 - V}{\omega_0^2} \frac{\omega_0}{\Omega} . \qquad (2.5)$$

Condition (1.21) here becomes

$$K = e^{-\delta} \alpha G^2 \omega / \Omega \gg 1 . \qquad (2.6)$$

But (2.2) and (2.3) indicate that (2.6) cannot be obeyed; for δ sufficiently small and in the limit, we get $K \sim 1$ from (2.3), which corresponds to a region intermediate between stability and stochastic behavior, so the motion may be said to be confused; it is difficult to examine.

Nearly stochastic motion may be obtained if we replace (2.2) by

$$0 < V - \omega_0^2 \ll \omega_0^2 ,$$

and hence (2.3) by

$$\alpha x^2 \omega_0^2 \ll V - \omega_0^2 . \qquad (2.7)$$

In this case there is a region where $\omega_n^2(t) < 0$, and the motion becomes unstable; the amplitude increases exponentially, and the change after passage through the region of ordinary instability may be sufficient to meet the SC, since matrix M in this case takes the form [12]

$$M \approx \left(\frac{\omega_{n+1}}{\omega_n}\right)^{1/2} \begin{pmatrix} \sqrt{1 + e^{2|\delta|}} & ie^{|\delta|} \\ -ie^{|\delta|} & \sqrt{1 + e^{2|\delta|}} \end{pmatrix},$$

in which δ is defined by (2.5); if we assume that $\delta \gtrsim 1$, the instability does not grow very rapidly, $\Delta x_n / x_n \sim \sim e^{|\delta|}$, and the SC takes the form

$$K = e^{|\delta|} \alpha G^2 \omega / \Omega \gg 1 . \qquad (2.8)$$

It is readily seen from (2.7) and (2.8) that, for δ not very large, the stochastic instability is of the same order as the instability in the region $\omega^2 < 0$, and (2.8) is not very strong ($K > 1$). The case $K \gg 1$ can be provided by large δ, when the increase in amplitude on account of passage through the region of dynamic instability is much greater than the increase on account of the stochastic instability.

In the case of (2.1), there is thus only a very small range of initial amplitudes in which stochastic instability can develop.

Stochastic instability in system (2.1) cannot develop for a long time, since the instability causes the nonlinearity to increase, and the system passes out of the region of strong resonance.

§ 3. External force with instantaneous action. The above method for the SC may be extended to systems different from (0.1). For instance, let the force F(t) of discontinuous type as in § 1 be external [5]:

$$x^{\cdot\cdot} + \omega_0^2 (1 + \alpha x^2) x = \omega_0 P \sum_k \delta(t - t_k) , \qquad (3.1)$$

in which the steps follow at equal intervals $T = 1/\Omega$. As in § 1, we have solutions as in (1.8) between two successive jumps t_k. There is the following relation between the amplitudes and phases at point t_n:

$$\frac{1}{\sqrt{\omega_n}} |A_n| \cos \varphi_n = \frac{1}{\sqrt{\omega_{n+1}}} |A_{n+1}| \cos(\varphi_{n+1} - S_{n+1}) ,$$

$$\operatorname{tg}(\varphi_{n+1} - S_{n+1}) = \frac{\omega_n}{\omega_{n+1}} \operatorname{tg} \varphi_n + \varepsilon_n,$$

$$\left| \frac{A_{n+1}}{A_n} \right|^2 =$$

$$= \frac{\omega_{n+1}}{\omega_n} \left[1 - \beta_n^- \sin^2 \varphi_n + 2\varepsilon_n^0 \frac{\omega_n}{\omega_{n+1}} \sin \varphi_n + (\varepsilon_n^0)^2 \right],$$

$$\varepsilon_n = \frac{\varepsilon_n^0}{\cos \varphi_n} , \qquad \varepsilon_n^0 = \frac{P}{A_n} \frac{\sqrt{\omega_n}}{\omega_{n+1}} . \qquad (3.2)$$

Formulas (3.2) show that the argument is similar to that of § 1, with substitution of ε_n in the matrix of

(1.9). Here ε_n is dependent on A_n and φ_n, so the matrix is nonlinear.

By analogy with (1.21) we have the SC as

$$K = \max\ (\varepsilon_n{}^0,\ (\varepsilon_n{}^0)^2)\ \alpha G^2\omega\ /\ \Omega \gg 1\ , \qquad (3.3)$$

or, substituting for ε_n^0 from (3.2),

$$K = \alpha GP\omega\ /\ \Omega \gg 1 \qquad (P \ll G),$$

$$K = \alpha P^2\omega\ /\ \Omega \gg 1 \qquad (P \gg G)\ . \qquad (3.4)$$

In the last inequality of (3.4), it is assumed that the condition of smallness of αx^2 in (3.1) is obeyed:

$$\alpha P^2 \ll 1\ .$$

§4. Conclusions. 1) The cases of §§1-3 show that the general basis for deriving the SC involves constructing a transition matrix corresponding to the time-shift operator and deriving an effective parameter ε characterizing the relative change in the adiabatic invariant. It is clear that it is easy to deduce the criterion for the case of a small nonlinearity of arbitrary type and for F(t) such that

$$F^{(n)}(t) = Q \sum_k \delta(t - t_k)', \qquad (4.1)$$

in which n is a derivative of any order.

2) In the case F(t) = A sin Ωt of §2, the role of F(t) can be played by another oscillator y whose inertia is so great that the influence of oscillator x on y can be neglected to a first order [8].

3) The above involves the major question of the time during which these results apply. The behavior can be described by a kinetic equation if the SC is obeyed; but the nonlinearity becomes great after the lapse of a large time, and other methods are required in order to examine the problem. The question of estimation of that time remains open.

I am indebted to R. Z. Sagdeev for directing my attention to the topic and to B. V. Chirikov for valuable advice and comments.

Appendix 1. Consider the influence on the solution of (2.4) from points t^* such that $\omega(t^*) = 0$. In accordance with (1.5) and (1.6), the t^* satisfy

$$1 + \alpha G^2 \cos^2\omega t^* = 0. \qquad (A.1.1)$$

Then the roots of (A.1.1) lie in the complex plane very far from the real axis and follow periodically (frequency ω) along that axis. The contribution of such singularities to change in the amplitude of the solution is exponentially small and is on the order of

$$\exp\left\{-\frac{1}{\beta}\right\} \ll 1, \qquad \beta = \frac{d\ln\omega}{dt}\left(\frac{d\ln x}{dt}\right)^{-1} \ll 1\ , \qquad (A.1.2)$$

in which β characterizes the slowness of the change. From (1.2) and (A.1.2) we get $\beta \approx (\alpha G^2)^{1/2}$, so the total amplitude change ΔG between two successive jumps is

$$\frac{\Delta G}{G} \sim \frac{\omega}{\Omega}\exp\left\{-\frac{1}{\sqrt{\alpha G^2}}\right\}\ . \qquad (A.1.3)$$

On the other hand, the change due to a jump is $\Delta G/G \sim \varepsilon = a/\omega$, i.e.,

$$\frac{\omega}{\Omega}\exp\left\{-\frac{1}{\sqrt{\alpha G^2}}\right\} \ll \frac{a}{\omega}\ . \qquad (A.1.4)$$

The left side is exponentially small, so this inequality is easily met at the same time as the SC of (1.21).

Appendix 2. Let some phase step of the oscillator be $\varphi_0 \ll K^{-1}$. Consider the time spent by the system in the region where the SC is not obeyed, i.e., $\varphi \lesssim K^{-1}$. As this region is small $K^{-1} \ll 1$), this estimate clearly applies also to (1.15) and (1.16), which for φ give

$$\varphi_1 = K\varphi_0, \quad \varphi_2 = K^2\varphi_0,\ldots, \quad \varphi(t) \sim K^{\Omega t}\varphi_0 \quad (\Omega t \gg 1)\ . \qquad (A.2.1)$$

Then (A.2.1) and the condition for attainment of the stochastic boundary give us the time t_0 spent in the region $\varphi_0 K \ll 1$:

$$t_0 \sim \frac{\ln\varphi_0^{-1}}{\Omega\ln K} \sim \tau_0 \ln\frac{1}{\varphi_0}\ . \qquad (A.2.2)$$

It follows from (A.2.2) and (1.22) that $t_0 \gg \tau_0$, so the probability of a fluctuation with $\varphi_0 \ll K^{-1}$ is exponentially small ($\sim\exp(-t_0/\tau_0)$). If $\varphi_0 \lesssim 1$, (A.2.1) shows that the system reaches the stochastic region after a few jumps.

The difference between (1.16) and (1.15) makes itself felt, in particular, in that in the first case there are three regions for which the probability of entry is exponentially small. A special study is required in order to evaluate the time spent by the system in the region where $|K\cos\varphi| < 1$.

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